



AN EXTENSION OF ZOLOTAREV'S PROBLEM AND SOME RELATED RESULTS *

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Abstract The main purpose of this paper is to extend the Zolotarev's problem concerning with geometric random sums to negative binomial random sums of independent identically distributed random variables. This extension is equivalent to describing all negative binomial infinitely divisible random variables and related results. Using Trotter-operator technique together with Zolotarev-distance's ideality, some upper bounds of convergence rates of normalized negative binomial random sums (in the sense of convergence in distribution) to Gamma, generalized Laplace and generalized Linnik random variables are established. The obtained results are extension and generalization of several known results related to geometric random sums.

Key words Zolotarev's problem; geometric random sum; negative binomial random sum; negative binomial infinitely divisibility; Trotter-operator technique

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1 Introduction

An analogue of Zolotarev's problem on geometric random sums originated by Klebanov *et al.* [12] is stated as follows: describe all random variables Y that for any $p \in (0, 1)$, there exists a random variable X_p such that

$$Y \stackrel{D}{=} X_p + \epsilon_p Y, \quad (1.1)$$

where ϵ_p is a Bernoulli random variable having probability mass function

$$\mathbb{P}(\epsilon_p = 0) = p \quad \text{and} \quad \mathbb{P}(\epsilon_p = 1) = 1 - p.$$

In eq. (1.1) the random variables Y , X_p and ϵ_p are independent. In eq. (1.1) and from now on, the notation $\stackrel{D}{=}$ expresses the equality in the sense of distributions.

The solution of eq. (1.1) is a geometric random sum, defined as follows

$$Y \stackrel{D}{=} \sum_{j=1}^{\nu_p} X_{p,j}, \quad (1.2)$$

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where $X_{p,j}, j \geq 1$ are independent, identically distributed (i.i.d.) random variables and ν_p is a geometric random variable with mean p^{-1} , ($0 < p < 1$), independent of all $X_{p,j}, j \geq 1$. Thus, Zolotarev's problem is equivalent to describing all geometrically infinitely divisible (see [12] for more details).

Geometric random summations arise in many applied problems in physics, biology, economics, insurance mathematics, reliability, regenerative models, etc. Up to now, geometric random summations have been investigated by many authors like Klebanov *et al.* [12], Kruglov and Korolev [20], Kalashnikov [10], Sandhya and Pillai [25], Asmussen [2], Gnedenko and Kruglov [5], Lin [21], Kotz *et al.* [16], Kozubowski [18], Bon [3], Gamboa and Pamphile [4], Malinowski [23], Aly and Bouzar [1], Hung [6], and Korolev *et al.* [13–15], etc.

It is worth pointing out that the convolution of k ($k \in \mathbb{N}$) geometric random variables with common parameter p , $p \in (0, 1)$ will be a negative binomial random variable with two parameters $p \in (0, 1)$ and $k \in \mathbb{N}$ (see for instance [27]). We propose here some natural enlargements of the class of geometrically infinitely divisible random variables is due to Klebanov *et al.* ([12], Definition 1, page 792). Let $N_{p,k}$ be a negative binomial distributed random variable with two parameters $p \in (0, 1)$ and $k \in \mathbb{N}$, denoted by $N_{p,k} \sim \text{NB}(p, k)$, whose probability mass function defined as follows

$$\mathbb{P}(N_{p,k} = x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}, \quad (1.3)$$

where x is an integer and $x \geq k$ (see [27], for more details). Recently, the negative binomial sums of i.i.d. random variables together with applications have been studied by Yakumiv [29], Sunklodas [27], Sheeja and Kumar [26], Janković [8], etc. It is clear that when $k = 1$, the $N_{p,1}$ reduces to a geometric random variable $\nu_p \sim \text{Geo}(p)$, $p \in (0, 1)$.

For any $p \in (0, 1)$ and $k \in \mathbb{N}$, let $\{X_{p,k,j}, j \geq 1\}$ be a sequence of random variables, independent of $N_{p,k}$. Consider the so-called negative binomial random sum

$$S_{N_{p,k}} = \sum_{j=1}^{N_{p,k}} X_{p,k,j}, \quad (1.4)$$

(see for instance [27]). It is plain that when $k = 1$, the negative binomial random sum in (1.4) reduces to the geometric random sum defined in (1.2).

By an argument analogous to result of Klebanov *et al.* [12], the negative binomial random sums in eq. (1.4) will be considered as an extension of Zolotarev's problem. This extension is equivalent to describing all negative binomial infinitely divisible random variables and related results. Using Trotter-operator technique [28] together with Zolotarev-distance's ideality, some upper bounds of convergence rates of normalized negative binomial random sums (in the sense of convergence in distribution) to Gamma, generalized Laplace and generalized Linnik random variables are established. Note that when $k = 1$, the well known results related to weak limit theorems for geometric random sums will be obtained as direct consequences of theorems in this article.

This paper is organized as follows. Section 2 presents an extension of Zolotarev's problem. Section 3 introduces the notion of a negative binomial infinitely divisible (NBID) random variable and related results. In Section 4, the accuracy of the approximations to the distributions of normalized negative binomial random sums are estimated.

2 An Extension of Zolotarev's Problem

Let ϵ_p be a Bernoulli random variable, defined in eq. (1.1). The problem to be considered in this section is that of describing all random variables $\mathbf{Y}$ that for any $p \in (0, 1)$ and $k \in \mathbb{N}$, there are random variables $X_{p,k}$ and Y_j such that

$$Y_j \stackrel{D}{=} X_{p,k} + \epsilon_p Y_j, \quad \text{for } j = 1, 2, \dots, k \quad (2.1)$$

and

$$\mathbf{Y} \stackrel{D}{=} \sum_{j=1}^k Y_j, \quad (2.2)$$

where random variables $X_{p,k}$, Y_j and ϵ_p are independent for $j = 1, 2, \dots, k$.

The following theorem will be considered as an extension of well-known result originated by Klebanov *et al.* in [12].

Theorem 2.1 Let assumptions (2.1) and (2.2) hold. Then, the random variable $\mathbf{Y}$ should be defined as a negative binomial random sum in the form

$$\mathbf{Y} \stackrel{D}{=} \sum_{j=1}^{N_{p,k}} X_{p,k,j}, \quad (2.3)$$

where $p \in (0, 1)$, $k \in \mathbb{N}$, $N_{p,k} \sim \text{NB}(p, k)$ and $X_{p,k,j}$ are i.i.d. random variables, independent of $N_{p,k}$ for $j \geq 1$.

Proof Let us denote by $f_{Y_j}(t)$, $f_{X_{p,k}}(t)$ and $f_{\mathbf{Y}}(t)$ the characteristic functions of random variables Y_j , $X_{p,k}$ and $\mathbf{Y}$, respectively. From (2.1), we have

$$\begin{aligned} f_{Y_j}(t) &= \mathbb{E} \left[e^{it(X_{p,k} + \epsilon_p Y_j)} \right] = \mathbb{E} \left(e^{itX_{p,k}} \right) \mathbb{E} \left[e^{it\epsilon_p Y_j} \right] = f_{X_{p,k}}(t) \mathbb{E} \left[e^{it(\epsilon_p Y_j)} \right] \\ &= f_{X_{p,k}}(t) \left[\mathbb{E} \left(e^{it(\epsilon_p=0)Y_j} \right) \mathbb{P}(\epsilon_p = 0) + \mathbb{E} \left(e^{it(\epsilon_p=1)Y_j} \right) \mathbb{P}(\epsilon_p = 1) \right] \\ &= f_{X_{p,k}}(t) \left[p + (1-p)f_{Y_j}(t) \right]. \end{aligned} \quad (2.4)$$

From (2.4) we conclude that

$$f_{Y_j}(t) = \frac{pf_{X_{p,k}}(t)}{1 - (1-p)f_{X_{p,k}}(t)}. \quad (2.5)$$

In the other hand, the relation (2.2) is equivalent to

$$f_{\mathbf{Y}}(t) = [f_{Y_j}(t)]^k.$$

Therefore, from (2.5) we have

$$f_{\mathbf{Y}}(t) = \left[\frac{pf_{X_{p,k}}(t)}{1 - (1-p)f_{X_{p,k}}(t)} \right]^k.$$

It follows that

$$\mathbf{Y} \stackrel{D}{=} \sum_{j=1}^{N_{p,k}} X_{p,k,j},$$

where $X_{p,k,j}$ are i.i.d. copied of $X_{p,k}$ and they are independent of $N_{p,k} \sim \text{NB}(p, k)$.

The proof is complete. $\square$

Remark 2.2 When $k = 1$, Theorem 2.1 reduces to Zolotarev's problem in [12].

3 Negative Binomial Infinitely Divisibility

The following notion will be needed in the sequel.

Definition 3.1 A random variable $\mathbf{Y}$ is said to be negative binomial infinitely divisible (NBID), if for any $p \in (0, 1)$ and for a fixed $k \in \mathbb{N}$, there exists a sequence of i.i.d. random variables $X_{p,k,1}, X_{p,k,2}, \dots$ such that

$$\mathbf{Y} \stackrel{D}{=} \sum_{j=1}^{N_{p,k}} X_{p,k,j},$$

where $N_{p,k} \sim \text{NB}(p, k)$, and $X_{p,k,1}, X_{p,k,2}, \dots$ are independent of $N_{p,k}$ and $\mathbf{Y}$.

Remark 3.2 The extension of Zolotarev's problem, formulated in Section 2 is thus equivalent to describing all negative binomial infinitely divisible random variables. For a deeper discussion of definition of negative binomial infinitely divisibility we refer the reader to [8] and [29].

Remark 3.3 When $k = 1$ the definition of NBID random variables reduces to the well known definition of geometric infinitely divisible (GID) random variables, originated by Klebanov *et al.* [12]. Results of this nature may be found in [1, 8, 16, 19, 21, 23].

The following theorem states that the weak limiting distribution of a negative binomial random summation is a NBID distribution. The symbol $\xrightarrow{D}$ hereinafter denotes convergence in distribution.

Theorem 3.4 For fixed $k \in \mathbb{N}$, let

$$\sum_{j=1}^{N_{p,k}} X_{p,k,j} \xrightarrow{D} \mathbf{Y}, \quad \text{as } p \rightarrow 0^+. \quad (3.1)$$

Then $\mathbf{Y}$ is a NBID random variable.

Proof Let us denote by $f_{X_{p,k}}(t)$ the common characteristic function of $X_{p,k,j}$ for $j \geq 1$. According to [7] (Theorem 9.1, page 193), the characteristic function of a negative-binomial random sum $S_{N_{p,k}} = \sum_{j=1}^{N_{p,k}} X_{p,k,j}$ is given by

$$f_{S_{N_{p,k}}}(t) = \left\{ \frac{p f_{X_{p,k}}(t)}{1 - (1-p) f_{X_{p,k}}(t)} \right\}^k.$$

Thus, for fixed k , from (3.1), it follows that

$$[f_k(t)]^k = f_{\mathbf{Y}}(t),$$

where $f_{\mathbf{Y}}(t)$ is characteristic function of the limit random variable $\mathbf{Y}$ and

$$f_k(t) = \lim_{p \rightarrow 0^+} \frac{p f_{X_{p,k}}(t)}{1 - (1-p) f_{X_{p,k}}(t)}.$$

It is clear that $f_k(t)$ is a characteristic function of limit distribution of a geometric random sum.

According to [20] (Theorem 8.1.2, page 243), the characteristic function $f(t)$ must be GID.

On the other hand, on account of Klebanov *et al.* [12] (Theorem 2, page 792), it follows that

$$f(t) = \exp \left\{ 1 - \frac{1}{f_k(t)} \right\}$$

is an infinitely divisible characteristic function. Therefore, we get

$$f_{\mathbf{Y}}(t) = [f_k(t)]^k = \left(\frac{1}{1 - \ln f(t)} \right)^k,$$

where $f(t)$ is an infinitely divisible characteristic function.

From Theorem 2.1 the proof is complete. $\square$

Remark 3.5 It is worth noticing that the weak limit distribution of a geometric random sum will be GID distribution (see, for instance, [20], Theorem 8.1.2, page 243). It was deduced from Theorem 3.4 for case of $k = 1$.

Theorem 3.6 A random variable $\mathbf{Y}$ is NBID if and only if its characteristic function $f_{\mathbf{Y}}(t)$ is given by

$$f_{\mathbf{Y}}(t) = \left(\frac{1}{1 - \ln \varphi(t)} \right)^k \quad \text{for } k \in \mathbb{N} \quad \text{and } t \in \mathbb{R}, \quad (3.2)$$

where $\varphi(t)$ is a characteristic function of some infinitely divisible (ID) distribution.

Proof We first observe that, on account of Theorem 2.1, the random variable $\mathbf{Y}$ is NBID if and only if both relations (2.1) and (2.2) hold. Evidently, the relation (2.1) holds if and only if all random variables $Y_j, j = 1, 2, \dots, k$ are geometrically infinitely divisible (see [12] for more details). According to Theorem 2 in [12], it follows that

$$f_{Y_j}(t) = \frac{1}{1 - \ln \varphi(t)} \quad \text{for } t \in \mathbb{R}, \quad (3.3)$$

where $\varphi(t)$ is an infinitely divisible characteristic function.

In the other hand, the relation (2.2) is equivalent to

$$f_{\mathbf{Y}}(t) = f_{Y_j}^k(t) \quad \text{for } k \in \mathbb{N} \quad \text{and } t \in \mathbb{R}.$$

According to (3.3), we conclude that, the random variable $\mathbf{Y}$ is NBID if and only if

$$f_{\mathbf{Y}}(t) = \left(\frac{1}{1 - \ln \varphi(t)} \right)^k \quad \text{for } k \in \mathbb{N} \quad \text{and } t \in \mathbb{R},$$

where $\varphi(t)$ is an infinitely divisible characteristic function. The proof is complete. $\square$

Remark 3.7 The similar result is due to Janković [8] and Yakumiv [29] with definition of negative binomial infinitely divisible random variable in the form

$$Y \stackrel{D}{=} \lim_{p \rightarrow 0} \sum_{j=1}^{\nu_p^r} X_p^j,$$

where ν_p^r is a random variable having negative binomial law with $p \in (0, 1), r \in \mathbb{R}^+, X_p^1, X_p^2, \dots$ are i.i.d. random variables, independent of Y and ν_p^r .

Remark 3.8 When $k = 1$, the relation (3.1) reduces to geometric infinitely divisible (GID) characteristic function that is

$$f_{\mathbf{Y}}(t) = \frac{1}{1 - \ln \varphi(t)} \quad \text{for } t \in \mathbb{R},$$

where $\varphi(t)$ is an infinitely divisible characteristic function (see [12], Theorem 2, page 792). Result of this nature also may be found in [20] (Theorem 8.1.1, pages 241–242).

Corollary 3.9 (The Lévy–Khinchin type representation) A random variable $\mathbf{Y}$ is NBID if and only if its characteristic function is given in the form

$$f_{\mathbf{Y}}(t) = \left\{ 1 - i\gamma t - \int_{-\infty}^{+\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x) \right\}^{-k} \quad \text{for } t \in \mathbb{R}, \quad (3.4)$$

where $k \in \mathbb{N}$ and γ is a real constant, $G(x)$ is a bounded non-decreasing function. Note that the function under the integral sign is equal to $-t^2/2$ at the point $x = 0$.

Proof According to Lévy–Khinchin formula (see [24], page 30, Theorem 1.16), the function $\varphi(t)$ is an infinitely divisible characteristic function if and only if it admits the representation

$$\varphi(t) = \exp \left\{ i\gamma t + \int_{-\infty}^{+\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x) \right\} \quad \text{for } t \in \mathbb{R}.$$

On account of Theorem 3.6, the random variable $\mathbf{Y}$ is NBID if and only if

$$f_{\mathbf{Y}}(t) = \left(\frac{1}{1 - \ln \varphi(t)} \right)^k \quad \text{for } k \in \mathbb{N}, t \in \mathbb{R}.$$

Equivalently,

$$f_{\mathbf{Y}}(t) = \left\{ 1 - i\gamma t - \int_{-\infty}^{+\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x) \right\}^{-k} \quad \text{for } t \in \mathbb{R}.$$

The proof is complete. $\square$

Remark 3.10 When $k = 1$, the relation (3.4) reduces to a geometric infinitely divisible (GID) characteristic function in form

$$f_{\mathbf{Y}}(t) = \left\{ 1 - i\gamma t - \int_{-\infty}^{+\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x) \right\}^{-1}, \quad \text{for } t \in \mathbb{R}.$$

(See [12], Corollary 1, page 792).

Some negative binomial infinitely divisible (NBID) random variables are shown as follows.

Example 3.11 Let $\mathbf{G}$ be a Gamma distributed random variable with two parameters $\lambda > 0$ and $k \in \mathbb{N}$, denoted by $\mathbf{G} \sim \text{Gamma}(\lambda, k)$, whose characteristic function is given by

$$f_{\mathbf{G}}(t) = \left(\frac{\lambda}{\lambda - it} \right)^k, \quad \text{for } t \in \mathbb{R}.$$

It is clear that $\mathbf{G}$ is a negative binomial infinitely divisible random variable. Indeed, we have

$$f_{\mathbf{G}}(t) = \left(\frac{1}{1 - \frac{it}{\lambda}} \right)^k = \left(\frac{1}{1 - \ln \varphi(t)} \right)^k,$$

where $\varphi(t) = \exp\left(\frac{it}{\lambda}\right)$. Hence, for $n \in \mathbb{N}$,

$$\varphi(t) = \exp\left\{\frac{it}{\lambda}\right\} = \left\{ \exp\left(\frac{it}{n\lambda}\right) \right\}^n, \quad \text{for } t \in \mathbb{R}.$$

Therefore, $\varphi(t)$ is an infinitely divisible characteristic function. By Theorem 3.4, we get the confirmation.

According to [16], a random variable L is said to be classical Laplace distributed random variable with parameters zero and $\sigma > 0$, denoted by $L \sim \text{Laplace}(0, \sigma)$, if its characteristic function has the form

$$f_L(t) = \frac{1}{1 + \sigma^2 t^2}, \quad \text{for } t \in \mathbb{R}.$$

Let $L_1, L_2, \dots$ be a sequence of independent, classical Laplace distributed random variables with parameters zero and $\sigma > 0$. For $k \in \mathbb{N}$, write

$$\mathcal{L} \stackrel{D}{=} L_1 + L_2 + \dots + L_k.$$

Then, the characteristic function of $\mathcal{L}$ is given by

$$f_{\mathcal{L}}(t) = \left(\frac{1}{1 + \sigma^2 t^2} \right)^k, \quad \text{for } t \in \mathbb{R}.$$

Extending the concept of classical Laplace distributed random variable, the generalized Laplace distributed random variable will be introduced as follows.

Definition 3.12 A random variable $\mathcal{L}$ is said to have generalized Laplace distribution, denoted by $\mathcal{L} \sim \text{GLaplace}(0, \sigma, k)$, if its characteristic function is defined by

$$f_{\mathcal{L}}(t) = \left(\frac{1}{1 + \sigma^2 t^2} \right)^k, \quad \text{for } t \in \mathbb{R}. \quad (3.5)$$

For a deeper discussion of the generalized Laplace distributions and their properties we refer the reader to [9, 11, 13, 16–18].

Remark 3.13 When $k = 1$, the generalized Laplace distributed random variable reduces to the classical Laplace distributed random variable with parameters zero and $\sigma > 0$.

Example 3.14 Let $\mathcal{L} \sim \text{GLaplace}(0, \sigma, k)$. Then, $\mathcal{L}$ is a NBID random variable. It is easily seen that for $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\varphi(t) = \exp \{ -\sigma^2 t^2 \} = \left\{ \exp \left(-\frac{\sigma^2 t^2}{n} \right) \right\}^n$$

is an infinitely divisible characteristic function. According to Theorem 3.6, we have the confirmation.

We follow the definition used in [16]. A random variable ξ is said to be symmetric Linnik distributed random variable with parameters $\alpha \in (0, 2]$ and $\sigma > 0$, denoted by $\xi \sim \text{Linnik}(\alpha, \sigma)$, if its characteristic function is given by

$$f_{\xi}(t) = \frac{1}{1 + \sigma^{\alpha} |t|^{\alpha}} \quad \text{for } t \in \mathbb{R}.$$

Lin [21] proved that Linnik distributions is geometric infinitely divisible (GID) and there are no closed-form expressions for the distribution and density function for Linnik random variable except for $\alpha = 2$, which corresponds to the Laplace distribution.

Let $\xi_1, \xi_2, \dots$ be a sequence of independent, symmetric Linnik distributed random variables with parameters $\alpha \in (0, 2]$ and $\sigma > 0$. For any $k \in \mathbb{N}$, set

$$\zeta \stackrel{D}{=} \xi_1 + \xi_2 + \dots + \xi_k.$$

Then, the characteristic function of ζ has following form

$$f_{\zeta}(t) = \left(\frac{1}{1 + \sigma^{\alpha} |t|^{\alpha}} \right)^k, \quad \text{for } t \in \mathbb{R}.$$

Extending the concept of symmetric Linnik distributed random variable, the generalized Linnik distributed random variable will be defined as follows.

Definition 3.15 A random variable ζ is said to have generalized Linnik distribution, denoted by $\zeta \sim \text{GLinnik}(\alpha, \sigma, k)$, if its characteristic function is given by

$$f_{\zeta}(t) = \left(\frac{1}{1 + \sigma^{\alpha}|t|^{\alpha}} \right)^k, \quad \text{for } t \in \mathbb{R}. \quad (3.6)$$

For a deeper discussion of generalized Linnik distributions we refer reader to [11, 13, 16–18], and the references given there.

Remark 3.16 When $k = 1$, the generalized Linnik distribution reduces to symmetric Linnik distribution. Moreover, for $k = 1$ and $\alpha = 2$, we obtain the classical Laplace distribution (see for instance [16]).

Example 3.17 Let $\zeta \sim \text{GLinnik}(\alpha, \sigma, k)$. Then, ζ is a NBID random variable.

For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$\varphi(t) = \exp(-\sigma^{\alpha}|t|^{\alpha}) = \left\{ \exp\left(-\frac{\sigma^{\alpha}|t|^{\alpha}}{n}\right) \right\}^n$$

is an infinitely divisible characteristic function. According to Theorem 3.6, we obtain the confirmation.

Theorem 3.18 Let $\{f_m(t), m = 1, 2, \dots\}$ be a sequence of NBID characteristic functions converging to some characteristic function $f(t)$. Then, $f(t)$ is a NBID characteristic function.

Proof Due to $\{f_m(t), m = 1, 2, \dots\}$ is a sequence of NBID characteristic functions, according to Theorem 3.6, there exists a sequence of infinitely divisible characteristic functions $\{\varphi_m(t), m = 1, 2, \dots\}$ such that

$$f_m(t) = \left(\frac{1}{1 - \ln \varphi_m(t)} \right)^k, \quad m = 1, 2, \dots$$

Letting $m \rightarrow \infty$, then $f_m(t)$ converges to the characteristic function $f(t)$, hence $\varphi_m(t)$ will converge to the characteristic function $\varphi(t)$, and

$$f(t) = \left(\frac{1}{1 - \ln \varphi(t)} \right)^k.$$

According to Lemma 1.20 ([24], page 30), $\varphi(t)$ is an infinitely divisible characteristic function. Therefore, on account of Theorem 3.6, it follows that $f(t)$ is a NBID characteristic function. This completes the proof. $\square$

Remark 3.19 When $k = 1$, the Theorem 3.6 deduces to GID characteristic function as a limiting characteristic function of sequence of GID characteristic functions (see, for example, [12], Theorem 1, page 792).

Theorem 3.20 Every NBID characteristic function is infinitely divisible.

Proof Let $f(t)$ be a arbitrary NBID characteristic function. According to Theorem 3.6, we have

$$f(t) = \left(\frac{1}{1 - \ln \varphi(t)} \right)^k,$$

where $\varphi(t)$ is an infinitely divisible characteristic function. It is plain that,

$$f(t) = \left(\frac{1}{1 + \psi(t)} \right)^k, \quad (3.7)$$

where $\varphi(t) = e^{-\psi(t)}$ is an infinitely divisible characteristic function. To complete the proof it remains to show that $f(t)$ is an infinitely divisible characteristic function.

First, we attempt to show that

$$h(t) = \frac{1}{1 + \psi(t)} \quad (3.8)$$

is an infinitely divisible characteristic function. According to Lukacs [22] (Theorem 12.2.3, page 320), for any $a > 1$ and $g(t)$ is an arbitrary characteristic function, then $\frac{a-1}{a-g(t)}$ is an infinitely divisible characteristic function. For $n \geq 1$, taking

$$a = 1 + \frac{1}{n} \quad \text{and} \quad g(t) = e^{-\frac{1}{n}\psi(t)},$$

we have

$$\frac{a-1}{a-g(t)} = \frac{\frac{1}{n}}{1 + \frac{1}{n} - e^{-\frac{1}{n}\psi(t)}} = \frac{1}{n \left[1 + \frac{1}{n} - e^{-\frac{1}{n}\psi(t)} \right]}.$$

On the other hand, by Taylor series expansion, we obtain

$$e^w = 1 + w + o(w),$$

where $\frac{o(w)}{w} \rightarrow 0$ as $w \rightarrow 0$. Hence,

$$e^{-\frac{1}{n}\psi(t)} = 1 - \frac{1}{n}\psi(t) + o\left(-\frac{1}{n}\psi(t)\right).$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{a-1}{a-g(t)} = \lim_{n \rightarrow \infty} \frac{1}{n \left[\frac{1}{n} + \frac{1}{n}\psi(t) + o\left(-\frac{1}{n}\psi(t)\right) \right]} = \frac{1}{1 + \psi(t)}.$$

On account of Petrov [24] (Lemma 1.20, page 30), the limit of a sequence of infinitely divisible characteristic functions is infinitely divisible. Therefore, the function $h(t)$ is defined by (3.8) is infinitely divisible. According to [22] (Theorem 5.3.2, page 109), the characteristic function $f(t)$ in (3.7), is the product of k infinitely divisible characteristic functions. Thus, $f(t)$ is an infinitely divisible characteristic function. The proof is complete. $\square$

Remark 3.21 It is to be noticed that every GID characteristic function is infinitely divisible (see, for instance, [20], page 242). It was deduced from Theorem 3.18 for $k = 1$.

4 Bounds of the Accuracy of the Approximations to the Distributions of Negative Binomial Random Sums

We follow the notation used in [28]. The symbol $C(\mathbb{R})$ will denote the set of all bounded uniformly continuous functions on $\mathbb{R}$ with norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$, and

$$C^r(\mathbb{R}) = \{f \in C(\mathbb{R}) : f^{(m)} \in C(\mathbb{R}), 1 \leq m \leq r\} \quad (r = 1, 2, \dots).$$

Lemma 4.1 Let $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ be two sequences of i.i.d. random variables with $\mathbb{E}|X_1|^r < +\infty$, $\mathbb{E}|Y_1|^r < +\infty$ ($r \in \mathbb{N}$). Assume that the following condition

$$\mathbb{E}(X_1^m) = \mathbb{E}(Y_1^m) \quad (4.1)$$

holds for $m = 1, 2, \dots, r-1$ ($r \in \mathbb{N}$). Suppose that N is a positive integer-valued random variable, independent of all X_j and Y_j ($j \geq 1$). Further, assume that $\mathbb{E}(N) < +\infty$. Then

$$\left| \mathbb{E}f\left(\sum_{j=1}^N X_j\right) - \mathbb{E}f\left(\sum_{j=1}^N Y_j\right) \right| \leq \mathbb{E}(N) \frac{\|f^{(r)}\|}{r!} (\mathbb{E}|X_1|^r + \mathbb{E}|Y_1|^r),$$

where $f \in C^r(\mathbb{R})$ and $\|f^{(r)}\| = \sup_{x \in \mathbb{R}} |f^{(r)}(x)|$.

Proof Using Taylor series expansion formula with Lagrange remainder for $f \in C^r(\mathbb{R})$, we can assert that

$$f(x) = f(0) + \sum_{m=1}^{r-1} \frac{f^{(m)}(0)}{m!} x^m + \frac{f^{(r)}(\theta x)}{r!} x^r,$$

where $\theta \in (0, 1)$ and $x \in \mathbb{R}$. From this, for X and Y are random variables and $f \in C^r(\mathbb{R})$, we conclude that

$$f(X) - f(Y) = \sum_{m=1}^{r-1} \frac{f^{(m)}(0)}{m!} (X^m - Y^m) + \frac{f^{(r)}(\theta X)}{r!} X^r - \frac{f^{(r)}(\theta Y)}{r!} Y^r.$$

Since $|f^{(r)}(x)| \leq \|f^{(r)}\|$ for any $x \in \mathbb{R}$, from (4.1) it follows that

$$\left| \mathbb{E}f(X) - \mathbb{E}f(Y) \right| \leq \frac{\|f^{(r)}\|}{r!} (\mathbb{E}|X|^r + \mathbb{E}|Y|^r). \quad (4.2)$$

Based on Trotter-operator's properties [28] and applying the inequality (4.2) to sequences $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ of i.i.d. random variables, we obtain

$$\left| \mathbb{E}f\left(\sum_{j=1}^n X_j\right) - \mathbb{E}f\left(\sum_{j=1}^n Y_j\right) \right| \leq n \left| \mathbb{E}f(X_1) - \mathbb{E}f(Y_1) \right| \leq n \frac{\|f^{(r)}\|}{r!} (\mathbb{E}|X_1|^r + \mathbb{E}|Y_1|^r).$$

Let N be a positive integer-valued random variable, independent of all X_j and Y_j for $j \geq 1$, with $\mathbb{E}(N) < +\infty$. Then

$$\begin{aligned} \left| \mathbb{E}f\left(\sum_{j=1}^N X_j\right) - \mathbb{E}f\left(\sum_{j=1}^N Y_j\right) \right| &= \sum_{n=1}^{\infty} \left\{ \mathbb{P}(N=n) \left| \mathbb{E}f\left(\sum_{j=1}^n X_j\right) - \mathbb{E}f\left(\sum_{j=1}^n Y_j\right) \right| \right\} \\ &\leq \sum_{n=1}^{\infty} \left\{ \mathbb{P}(N=n) n \frac{\|f^{(r)}\|}{r!} (\mathbb{E}|X_1|^r + \mathbb{E}|Y_1|^r) \right\} \\ &= \mathbb{E}(N) \frac{\|f^{(r)}\|}{r!} (\mathbb{E}|X_1|^r + \mathbb{E}|Y_1|^r). \end{aligned}$$

The proof is complete. $\square$

Theorem 4.2 Let $X_1, X_2, \dots$ be a sequence of non-negative i.i.d. random variables with positive mean $\mathbb{E}X_1 = \mu > 0$ and finite variance $0 < \text{Var}(X_1) = \sigma^2 < +\infty$. Assume that $\mathbf{G} \sim \text{Gamma}(k, k\mu^{-1})$ and $N_{p,k} \sim \text{NB}(p, k)$, independent of all X_j for $j \geq 1$. Then

$$\left| \mathbb{E}f\left((p/k) \sum_{j=1}^{N_{p,k}} X_j\right) - \mathbb{E}f(\mathbf{G}) \right| \leq (p/k) \frac{\|f''\|}{2} (3\mu^2 + \sigma^2),$$

where $f \in C^2(\mathbb{R})$.

Proof According to Example 3.11, the $\mathbf{G}$ is a NBID random variable. Let $\{Z_j, j \geq 1\}$ be a sequence of independent, exponential distributed random variables with parameter μ^{-1} , independent of $N_{p,k}$. It is easily seen that the characteristic function of random variables Z_j is given by

$$\varphi_{Z_1}(t) = \frac{\mu^{-1}}{\mu^{-1} - it}, \quad t \in \mathbb{R}.$$

According to [7] (Theorem 9.1, page 193), the characteristic function of the negative-binomial sums $(p/k) \sum_{j=1}^{N_{p,k}} Z_j$ is defined by

$$\varphi_{(p/k) \sum_{j=1}^{N_{p,k}} Z_j}(t) = \left\{ \frac{\frac{p\mu^{-1}}{\mu^{-1} - i(p/k)t}}{1 - (1-p)\frac{\mu^{-1}}{\mu^{-1} - i(p/k)t}} \right\}^k = \left\{ \frac{k\mu^{-1}}{k\mu^{-1} - it} \right\}^k = \varphi_{\mathbf{G}}(t).$$

On account of the continuity theorem ([7], Theorem 9.1, page 257), the Gamma distributed random variable $\mathbf{G}$ admits the following presentation

$$\mathbf{G} \stackrel{D}{=} (p/k) \sum_{j=1}^{N_{p,k}} Z_j.$$

It is immediate that

$$\mathbb{E}Z_1 = \mathbb{E}X_1 = \mu; \quad \mathbb{E}(Z_1^2) = 2\mu^2 \quad \text{and} \quad \mathbb{E}(X_1^2) = \mu^2 + \sigma^2.$$

Applying Lemma 4.1 for $r = 2$, we obtain

$$\begin{aligned} \left| \mathbb{E}f\left((p/k) \sum_{j=1}^{N_{p,k}} X_j\right) - \mathbb{E}f(\mathbf{G}) \right| &= \left| \mathbb{E}f\left((p/k) \sum_{j=1}^{N_{p,k}} X_j\right) - \mathbb{E}f\left((p/k) \sum_{j=1}^{N_{p,k}} Z_j\right) \right| \\ &\leq \mathbb{E}(N_{p,k}) \frac{\|f''\|}{2} \left(\mathbb{E}|(p/k)X_1|^2 + \mathbb{E}|(p/k)Y_1|^2 \right) \\ &= (p/k) \frac{\|f''\|}{2} (3\mu^2 + \sigma^2). \end{aligned}$$

This completes the proof. $\square$

Remark 4.3 The following results are direct consequences of Theorem 4.2.

1. Let $k \in \mathbb{N}$ be a fixed number. A limit theorem for negative binomial sums of i.i.d. random variables is stated as follows

$$(p/k) \sum_{j=1}^{N_{p,k}} X_j \xrightarrow{D} \mathbf{G} \quad \text{as } p \rightarrow 0^+.$$

Here and subsequently, the symbol $\xrightarrow{D}$ stands for the convergence in distribution.

2. When $k = 1$, the Gamma distributed random variable $\mathbf{G}$ reduces to $\mathbf{E}_\mu \sim \text{Exp}(\mu^{-1})$, an exponential distributed random variable with mean $\mu > 0$. Then

$$\left| \mathbb{E}f\left(p \sum_{j=1}^{\nu_p} X_j\right) - \mathbb{E}f(\mathbf{E}_\mu) \right| \leq p \frac{\|f''\|}{2} (3\mu^2 + \sigma^2),$$

where $\nu_p \sim \text{Geo}(p)$. Moreover, the Rényi's limit theorem for geometric sums of i.i.d. random variables (see e.g [20], Theorem 8.1.5, page 246) is stated as follows

$$p \sum_{j=1}^{\nu_p} X_j \xrightarrow{D} \mathbf{E}_\mu \quad \text{as } p \rightarrow 0^+.$$

Theorem 4.4 Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}X_1 = 0$, $\mathbb{E}(X_1^2) = \sigma^2$ and $\mathbb{E}|X_1|^3 = \varrho < +\infty$. Let $N_{p,k} \sim \text{NB}(p, k)$, independent of all X_j , $j \geq 1$. Assume that $\mathcal{L} \sim \text{GLaplace}(0, \sigma/\sqrt{2k}, k)$ with $\sigma > 0$. Then

$$\left| \mathbb{E}f\left((p/k)^{1/2} \sum_{j=1}^{N_{p,k}} X_j\right) - \mathbb{E}f(\mathcal{L}) \right| \leq (p/k)^{1/2} \frac{\|f'''\|}{6} \left(\varrho + \frac{3\sigma^3}{\sqrt{2}} \right),$$

where $f \in C^3(\mathbb{R})$.

Proof On account of Example 3.14, $\mathcal{L} \sim \text{GLaplace}(0, \sigma/\sqrt{2k}, k)$ is a NBID random variable. Analysis similar to that in the proof of Theorem 4.2 shows that

$$\mathcal{L} \stackrel{D}{=} (p/k)^{1/2} \sum_{j=1}^{N_{p,k}} L_j,$$

where $\{L_j, j \geq 1\}$ is a sequence of independent identically Laplace distributed random variables, independent of $N_{p,k}$. It is clear that the characteristic function of L_j is given by

$$\varphi_{L_1}(t) = \frac{1}{1 + \frac{\sigma^2}{2}t^2}, \quad t \in \mathbb{R}.$$

According to Kotz *et al.* ([16], page 20), since $L_1 \sim \text{Laplace}(0, \sigma/\sqrt{2})$, it is a simple matter to

$$\begin{aligned} \mathbb{E}L_1 &= \mathbb{E}X_1 = 0; \quad \mathbb{E}(L_1^2) = \mathbb{E}(X_1^2) = \sigma^2 \\ \mathbb{E}|L_1|^3 &= \frac{3\sigma^3}{\sqrt{2}}; \quad \mathbb{E}|X_1|^3 = \varrho. \end{aligned}$$

Applying Lemma 4.1 for $r = 3$, we obtain

$$\begin{aligned} \left| \mathbb{E}f\left((p/k)^{1/2} \sum_{j=1}^{N_{p,k}} X_j\right) - \mathbb{E}f(\mathcal{L}) \right| &= \left| \mathbb{E}f\left((p/k)^{1/2} \sum_{j=1}^{N_{p,k}} X_j\right) - \mathbb{E}f\left((p/k)^{1/2} \sum_{j=1}^{N_{p,k}} L_j\right) \right| \\ &\leq \mathbb{E}(N_{p,k}) \frac{\|f'''\|}{3!} \left(\mathbb{E}|(p/k)^{1/2} X_1|^3 + \mathbb{E}|(p/k)^{1/2} Y_1|^3 \right) \\ &= (p/k)^{1/2} \frac{\|f'''\|}{6} \left(\varrho + \frac{3\sigma^3}{\sqrt{2}} \right). \end{aligned}$$

The proof is complete. $\square$

Remark 4.5 As immediate consequences of Theorem 4.4, we have

1. For fixed $k \in \mathbb{N}$ the weak limit theorem for negative binomial sums of i.i.d. random variables is stated as follows

$$(p/k)^{1/2} \sum_{j=1}^{N_{p,k}} X_j \xrightarrow{D} \mathcal{L} \quad \text{as } p \rightarrow 0^+.$$

2. When $k = 1$, then

$$\left| \mathbb{E}f\left(p^{1/2} \sum_{j=1}^{\nu_p} X_j\right) - \mathbb{E}f(L) \right| \leq p^{1/2} \frac{\|f'''\|}{6} \left(\varrho + \frac{3\sigma^3}{\sqrt{2}} \right),$$

where $L \sim \text{Laplace}(0, \sigma/\sqrt{2})$ and $\nu_p \sim \text{Geo}(p)$. Moreover, the weak limit theorem for geometric sums of i.i.d. random variables is formulated in the form

$$p^{1/2} \sum_{j=1}^{\nu_p} X_j \xrightarrow{D} L \quad \text{as } p \rightarrow 0^+.$$

It is worth pointing out that the Lemma 4.1 could not apply to the negative binomial sums $(p/k)^{1/\alpha} \sum_{j=1}^{N_{p,k}} X_j$ ($0 < \alpha \leq 2$), because its weak limiting distribution is Linnik law, whose absolute moments of order r are finite for $0 < r < \alpha$ and infinite for $r \geq \alpha$ (see [16], page 212). Therefore, in the sequel, we consider the subset of $C^1(\mathbb{R})$ is defined by

$$\mathfrak{F} = \{f \in C^1(\mathbb{R}) : |f'(x) - f'(y)| \leq |x - y|\},$$

where $x, y \in \mathbb{R}$. For any $f \in \mathfrak{F}$, we have the following lemma.

Lemma 4.6 Let $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ be two sequences of i.i.d. random variables with $\mathbb{E}|X_1| < +\infty$ and $\mathbb{E}|Y_1| < +\infty$. Assume that N is a positive integer-valued random variable, independent of all X_j and Y_j for $j \geq 1$. Assume that $\mathbb{E}(N) < +\infty$. Then, for any $c \neq 0$ and $f \in \mathfrak{F}$, there exists a positive constant M such that

$$\left| \mathbb{E}f\left(c \sum_{j=1}^N X_j\right) - \mathbb{E}f\left(c \sum_{j=1}^N Y_j\right) \right| \leq c^2 \mathbb{E}(N) M (\mathbb{E}|X_1| + \mathbb{E}|Y_1|).$$

Proof Since $f \in \mathfrak{F}$, one has by the Taylor series expansion

$$f(x) = f(0) + xf'(\eta x),$$

where $0 < \eta < 1$ and $x \in \mathbb{R}$. Since $|f'(x)| \leq \|f'\|$ for any $x \in \mathbb{R}$ and $f \in \mathfrak{F}$, for random variables X and Y with $\mathbb{E}|X| < +\infty$, $\mathbb{E}|Y| < +\infty$, we have

$$|\mathbb{E}f(X) - \mathbb{E}f(Y)| \leq \|f'\| (\mathbb{E}|X| + \mathbb{E}|Y|). \quad (4.3)$$

According to Zolotarev-distance's ideality (see [30] for more details), from (4.3), we obtain

$$\begin{aligned} \left| \mathbb{E}f\left(c \sum_{j=1}^n X_j\right) - \mathbb{E}f\left(c \sum_{j=1}^n Y_j\right) \right| &\leq \sup_{f \in \mathfrak{F}} \left| \mathbb{E}f\left(c \sum_{j=1}^n X_j\right) - \mathbb{E}f\left(c \sum_{j=1}^n Y_j\right) \right| \\ &\leq c^2 n \sup_{f \in \mathfrak{F}} |\mathbb{E}f(X_1) - \mathbb{E}f(Y_1)| \\ &\leq c^2 n \sup_{f \in \mathfrak{F}} \left\{ \|f'\| (\mathbb{E}|X_1| + \mathbb{E}|Y_1|) \right\}. \end{aligned}$$

Taking $M = \sup_{f \in \mathfrak{F}} \{\|f'\|\}$, we can assert that

$$\left| \mathbb{E}f\left(c \sum_{j=1}^n X_j\right) - \mathbb{E}f\left(c \sum_{j=1}^n Y_j\right) \right| \leq c^2 n M (\mathbb{E}|X_1| + \mathbb{E}|Y_1|). \quad (4.4)$$

From (4.4) we conclude that

$$\left| \mathbb{E}f\left(c \sum_{j=1}^N X_j\right) - \mathbb{E}f\left(c \sum_{j=1}^N Y_j\right) \right| = \sum_{n=1}^{\infty} \left\{ \mathbb{P}(N=n) \left| \mathbb{E}f\left(c \sum_{j=1}^n X_j\right) - \mathbb{E}f\left(c \sum_{j=1}^n Y_j\right) \right| \right\}$$

$$\begin{aligned} &\leq \sum_{n=1}^{\infty} \left\{ \mathbb{P}(N=n) c^2 n M \left(\mathbb{E}|X_1| + \mathbb{E}|Y_1| \right) \right\} \\ &= c^2 \mathbb{E}(N) M \left(\mathbb{E}|X_1| + \mathbb{E}|Y_1| \right). \end{aligned}$$

This completes the proof. $\square$

Theorem 4.7 Let $\{X_j, j \geq 1\}$ be a sequence of i.i.d. random variables with $\mathbb{E}|X_1| = \rho < +\infty$. Assume that $N_{p,k} \sim \text{NB}(p, k)$, independent of all X_j for $j \geq 1$. Then, for any $f \in \mathfrak{F}$, there exists a positive constant M such that

$$\left| \mathbb{E}f \left((p/k)^{1/\alpha} \sum_{j=1}^{N_{p,k}} X_j \right) - \mathbb{E}f(\zeta) \right| \leq (p/k)^{\frac{2-\alpha}{\alpha}} M \left(\rho + \frac{2\sigma}{\alpha \sin \frac{\pi}{\alpha}} \right),$$

where $\zeta \sim \text{GLinnik}(\alpha, \sigma k^{-1/\alpha}, k)$ and $1 < \alpha < 2$.

Proof On account of Example 3.17, $\zeta \sim \text{GLinnik}(\alpha, \sigma k^{-1/\alpha}, k)$ is a NBID random variable. Let $\{\xi_j, j \geq 1\}$ be a sequence of independent, Linnik distributed random variables with two parameters $\sigma > 0$ and $\alpha \in (0, 2]$. It is obvious that the characteristic function of ξ_j is given by

$$\varphi_{\xi_1}(t) = \frac{1}{1 + \sigma^\alpha |t|^\alpha}, \quad t \in \mathbb{R}.$$

Assume that $N_{p,k}$ is independent of all ξ_j for $j \geq 1$. By an analogous to Theorem 4.2 we have

$$\zeta \stackrel{D}{=} (p/k)^{1/\alpha} \sum_{j=1}^{N_{p,k}} \xi_j.$$

According to Kotz *et al.* ([16], Proposition 4.3.18, p. 212), since $\xi_1 \sim \text{Linnik}(\alpha, \sigma)$, we have

$$\mathbb{E}|\xi_1| = \frac{2\sigma}{\alpha \sin \frac{\pi}{\alpha}}, \quad 1 < \alpha < 2.$$

Applying Lemma 4.6 to the case $c = p/k$, we obtain

$$\begin{aligned} \left| \mathbb{E}f \left((p/k)^{1/\alpha} \sum_{j=1}^{N_{p,k}} X_j \right) - \mathbb{E}f(\zeta) \right| &= \left| \mathbb{E}f \left((p/k)^{1/\alpha} \sum_{j=1}^{N_{p,k}} X_j \right) - \mathbb{E}f \left((p/k)^{1/\alpha} \sum_{j=1}^{N_{p,k}} \xi_j \right) \right| \\ &\leq (p/k)^{2/\alpha} \mathbb{E}(N_{p,k}) M \left(\mathbb{E}|X_1| + \mathbb{E}|\xi_1| \right) \\ &\leq (p/k)^{\frac{2-\alpha}{\alpha}} M \left(\rho + \frac{2\sigma}{\alpha \sin \frac{\pi}{\alpha}} \right). \end{aligned}$$

The proof is complete. $\square$

Remark 4.8 Under the assumptions of Theorem 4.7 it follows that

1. Let $k \in \mathbb{N}$ be a fixed. Then

$$(p/k)^{1/\alpha} \sum_{j=1}^{N_{p,k}} X_j \xrightarrow{D} \zeta \quad \text{as } p \rightarrow 0^+.$$

2. When $k = 1$ we have

$$\left| \mathbb{E}f \left(p^{1/\alpha} \sum_{j=1}^{\nu_p} X_j \right) - \mathbb{E}f(\xi) \right| \leq p^{\frac{2-\alpha}{\alpha}} M \left(\rho + \frac{2\sigma}{\alpha \sin \frac{\pi}{\alpha}} \right).$$

Moreover, the weak limit theorem for geometric random sum is stated as follows

$$p^{1/\alpha} \sum_{j=1}^{\nu_p} X_j \xrightarrow{D} \xi \quad \text{as } p \rightarrow 0^+.$$

Concluding remarks We conclude this paper with the following comments.

1. In this paper, an extension of Zolotarev's problem in [12] is discussed. The solution of the considered is a so-called negative binomial random sum that is equivalent to describing of negative binomial infinitely divisible (NBID) random variables. Some well known NBID distributions like Gamma, generalized Laplace and generalized Linnik distributions are shown as research examples. Estimate of convergence rates of normalized negative binomial random sums (in the sense of convergence in distribution) to Gamma, generalized Laplace and generalized Linnik random variables is also research objective of this article. The obtained results are extension and generalization of several known ones related to geometric random summations of i.i.d. random variables. The mathematical tool used in study of estimates of convergence rates is Trotter-operator technique together with Zolotarev-distance's ideality. Some analogous notions and definitions in this paper may be found in references [13–18].

2. An analogous result for the first point of Remark 4.7 is due to Korolev *et al.* in [13], using different way of proving. Theorem 6 ([13], page 13) is stated that

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{N_{n,\nu}} X_j \xrightarrow{D} \zeta_{\alpha,\nu} \quad \text{as } n \rightarrow \infty,$$

where $N_{n,\nu} \sim \text{NB}(1/n, \nu)$, $\nu > 0$, independent of X_j for $j \geq 1$. Note that the first parameter p of $\text{NB}(p, \nu)$ is taken as $p = 1/n$ and $p \rightarrow 0^+$ as $n \rightarrow \infty$.

3. The necessary and sufficient conditions of the convergence of distributions of random sums of i.i.d. random variables with finite variance to the Linnik distribution may be found in [14] (Theorem 4, page 13) and [15] (Theorem 4, page 9).

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